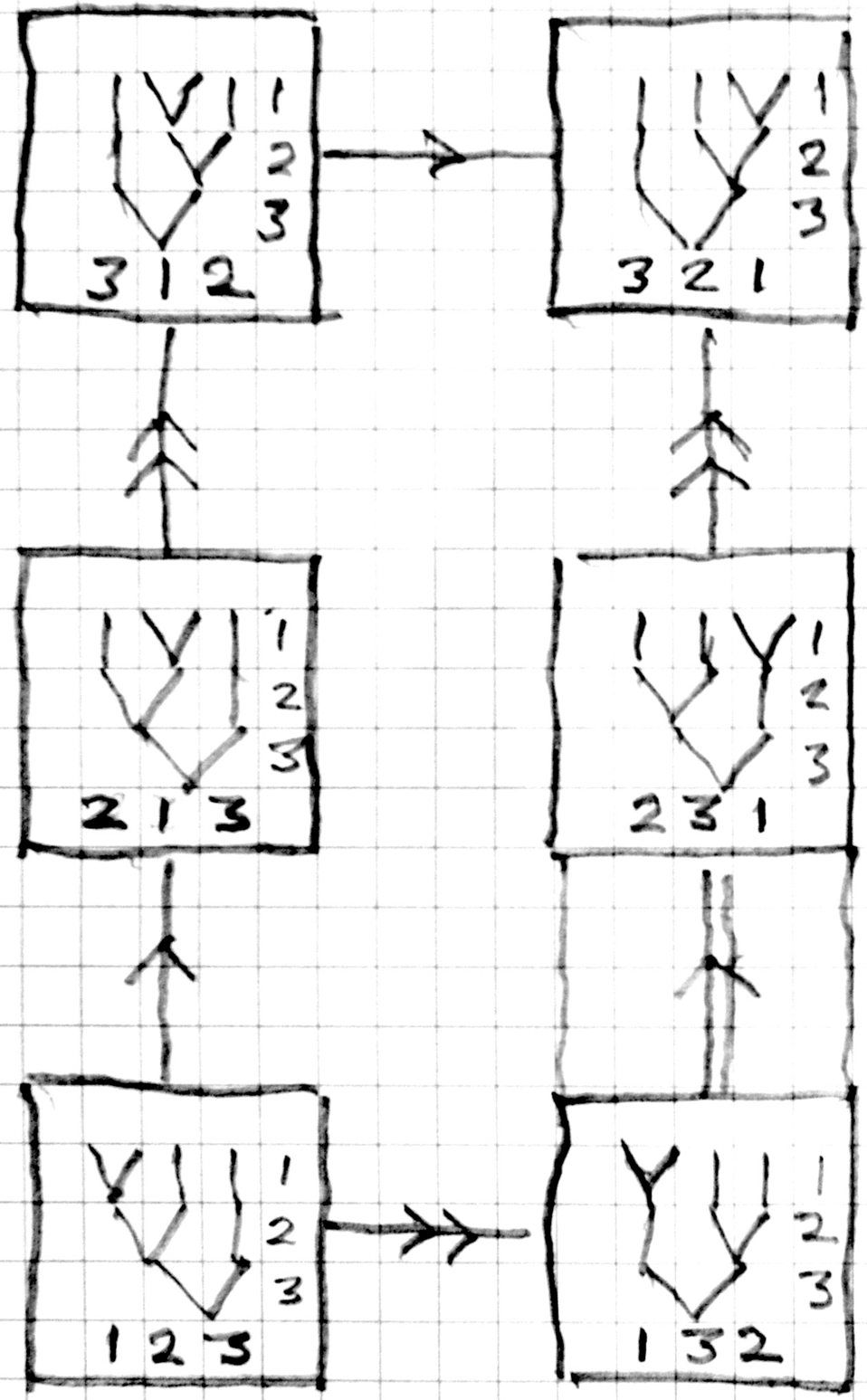


$$\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \quad (123)$$

$$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \quad (1\ 3)$$

$$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \quad (1\ 2)$$

$$\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \quad (1\ 3\ 2)$$

1 goes in 3rd,
2 goes in 1st,
3 goes in 2nd
place.

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad (2\ 3)$$

$$\begin{bmatrix} 1/2 & 1/2 & \\ 1/2 & 1/2 & \\ & & 1 \end{bmatrix}$$

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "id\otimes m"] \arrow[d, "m\otimes id"'] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
 & A\otimes A \arrow[d, "m"] & \\
A\otimes k \arrow[ur, "id\otimes\eta"] \arrow[r, equal, "\cong"] & A \arrow[r, equal, "\cong"] & k\otimes A \arrow[ul, "\eta\otimes id"']
\end{tikzcd}$$

ALGEBRA

$\Updownarrow$ DUALIZE $\Updownarrow$

$$\begin{tikzcd}
A\otimes A\otimes A & A\otimes A \arrow[l, "id\otimes\Delta"'] \\
A\otimes A \arrow[u, "\Delta\otimes id"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & A\otimes A \arrow[dl, "id\otimes\varepsilon"'] \arrow[d, "\Delta"] \arrow[dr, "\varepsilon\otimes id"] & \\
A\otimes k \arrow[r, equal, "\cong"] & A \arrow[r, equal, "\cong"] & k\otimes A
\end{tikzcd}$$

COALGEBRA

COMBINE WITH BELOW FOR HOPF ALGEBRA

$$\begin{tikzcd}
H\otimes H \arrow[d, "id\otimes S, S\otimes id"'] & H \arrow[l, "\Delta"'] \arrow[d, "\varepsilon"] & H\otimes H \arrow[l, "m"'] \arrow[dl, "\varepsilon\otimes\varepsilon"] \\
 & k \arrow[d, "\eta"] \arrow[dr, "\eta\otimes\eta"] & \\
H\otimes H \arrow[r, "m"'] & H \arrow[r, "\Delta"'] & H\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
H\otimes H\otimes H\otimes H \arrow[d, "id\otimes\tau\otimes id"'] & H\otimes H \arrow[l, "\Delta\otimes\Delta"'] \arrow[d, "m"] \\
 & H \arrow[d, "\Delta"] \\
H\otimes H\otimes H\otimes H \arrow[r, "m\otimes m"'] & H\otimes H
\end{tikzcd}$$

(32514)

$$\begin{bmatrix} & & & 1 & \\ & & & & 1 \\ & 1 & & & \\ & & 1 & & \\ 1 & & & & \end{bmatrix}$$

(15234)

Sweep the first three left rows.

Sweep the last two rows right

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

$(123) \otimes (12)$.

$(132) \otimes (12)$

right

left

Comultiplication on $\bigoplus_{n \in \mathbb{N}} k S_n$

Δ

assigns sum of all these shuffles.

$m : ((123), (12)) \longmapsto (123)(45)$

$S_r \times S_s \longrightarrow S_{r+s}$

$(\rho, \sigma) \longmapsto \rho \oplus \sigma$

ρ on first r, σ on last s

Hopf algebra

$\bigoplus_{n \in \mathbb{N}} k S_n$

J.-L. Loday & M.O. Ronco, Hopf Algebra of the Planar Binary Trees, Adv. Math. 139 (1998), 293-309.

Y_n : set of binary trees with n leaves

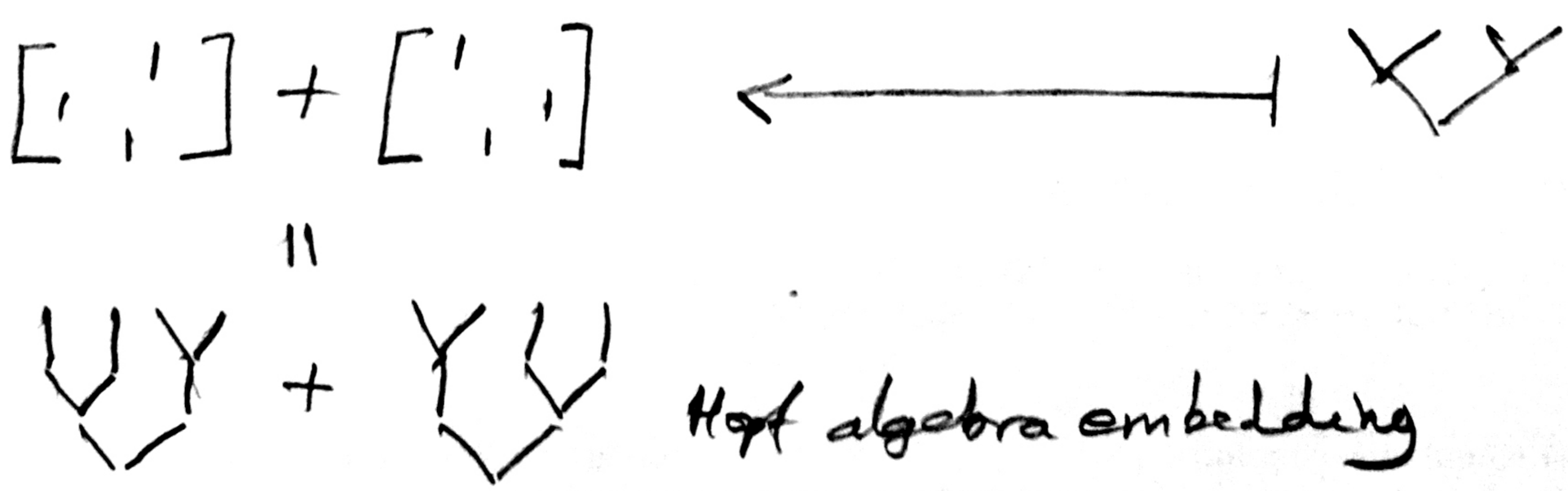
$\tilde{Y}_n$: ... levelled ...

$$\bigoplus_{n\in\mathbb{N}} k S_n \cong \bigoplus_{n\in\mathbb{N}} k \tilde{Y}_n \xrightarrow{\text{forget levels}} \bigoplus_{n\in\mathbb{N}} k Y_n$$

$$\bigoplus_{n\in\mathbb{N}} k S_n \xrightarrow{\psi} \bigoplus_{n\in\mathbb{N}} k Y_n$$

has linear dual

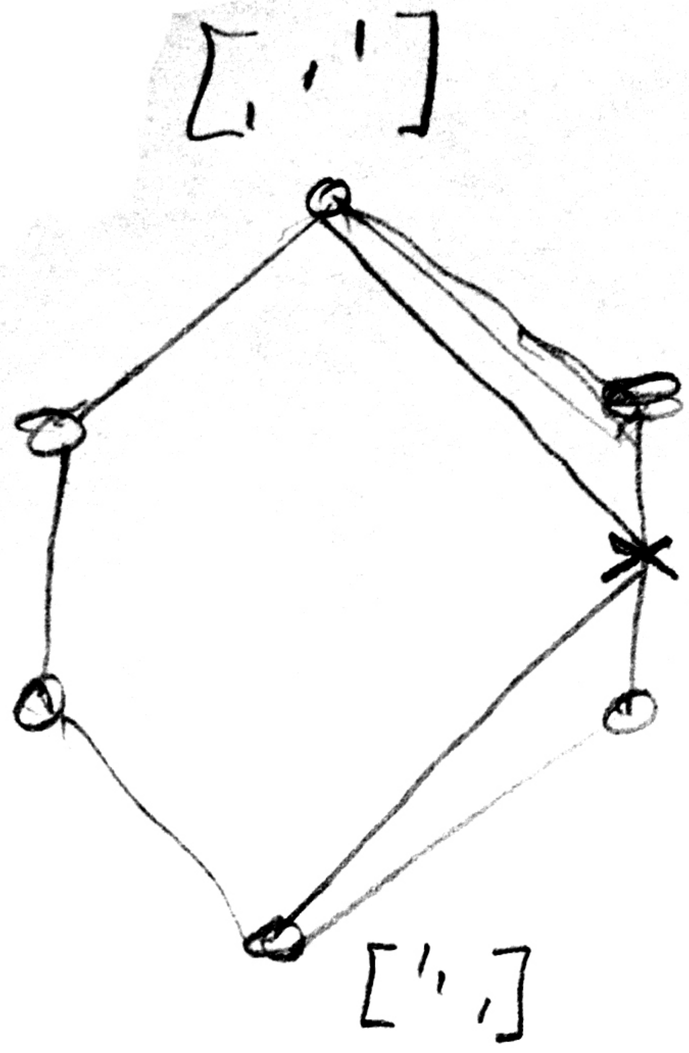
$$\bigoplus_{n\in\mathbb{N}} k S_n \xleftarrow{\psi^*} \bigoplus_{n\in\mathbb{N}} k Y_n$$



Hopf algebra embedding

Lattice embedding

Barycentric alg. embedding



Check geometry: are the 3x3 perm. matrices coplanar? [If so, eqn. of plane in 9-dim. space $\mathbb{R}^3_3$.]

Not coplanar:

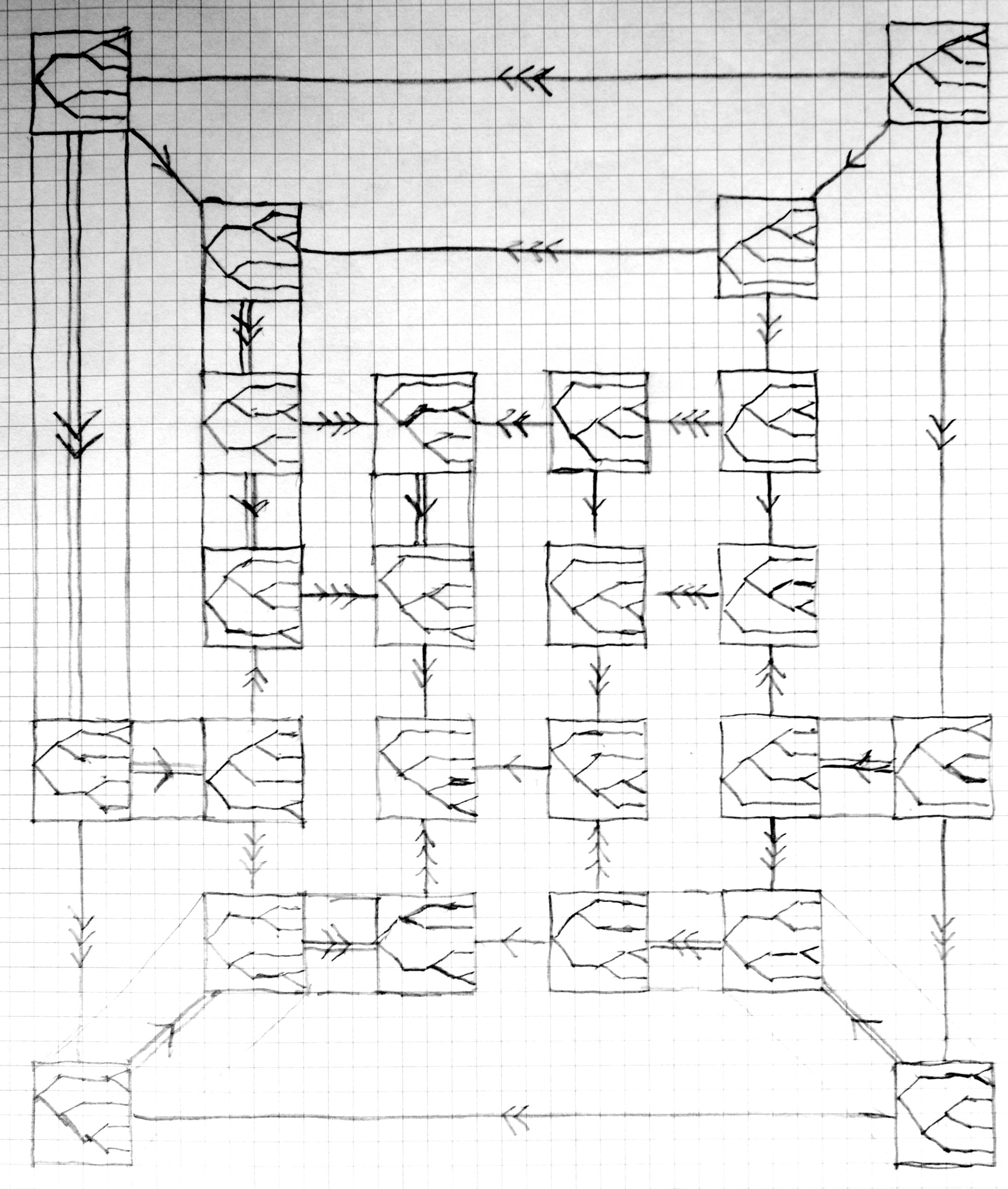
line $\begin{bmatrix} 0 & 1-t & t \\ 0 & t & 1-t \\ 1 & 0 & 0 \end{bmatrix}$ joining $\overset{(123)}{\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}}$ to $\overset{(13)}{\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}}$

does not intersect

line $\begin{bmatrix} 1 & & \\ & 1-t & t \\ & t & 1-t \end{bmatrix}$ joining $\overset{(1)}{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}$ to $\overset{(23)}{\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}}$

Since $n^2 < n!$, barycentric alg. spanned by S_n in $\mathbb{R}^n_n$ will not be free.

¿ But still projects to permutohedron?

Permanent of $n \times n$ (0-1)-matrix X counts number of $n \times n$ perm. matrices that embed into X.

— relate to entropy in permutohedron?